

Some abstract results on the existence of bounded Palais-Smale sequences

Michela Guida, Sergio Rolando

Dipartimento di Matematica “Giuseppe Peano”

Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

e-mail: michela.guida@unito.it, sergio.rolando@unito.it

Abstract

Without compactness assumptions, we prove some abstract results which show that a C^1 functional $I : X \rightarrow \mathbb{R}$ on a Banach space X admits bounded Palais-Smale sequences provided that it exhibits some geometric structure of minimax type and a suitable behaviour with respect to some sequence of continuous mappings $\psi_n : X \rightarrow X$. This work is a preliminary version of a forthcoming paper, where applications to nonlinear equations without Ambrosetti-Rabinowitz type assumptions will also be given.

1 Introduction

This paper is concerned with the existence of bounded Palais-Smale sequences for C^1 functionals defined on a Banach space and satisfying suitable minimax conditions. This kind of study is a fundamental step in minimax methods of critical point theory, where existence of critical points for a given functional $I : X \rightarrow \mathbb{R}$ on a Banach space X is usually obtained by searching for sequences of “almost critical” points and then showing their convergence to an exact critical point. More precisely, a minimax method commonly conforms to the following scheme (see [8], [20]):

- a geometric condition is required, involving a relation between the values of I over sets that satisfy a topological intersection property (*linking property*);
- by a quantitative deformation lemma (or by Ekeland’s variational principle), the existence of a *Palais-Smale sequence* for I is provided, namely, a sequence $\{w_n\} \subset X$ such that

$$I(w_n) \rightarrow c \quad \text{and} \quad I'(w_n) \rightarrow 0 \text{ in } X'$$

where c is some real value defined by a minimax procedure (and X' is the dual space of X);

- some *a posteriori* compactness property of $\{w_n\}$ is proved, with a view to concluding that the “almost critical” points w_n actually approximate an exact critical point of I .

A wide range of abstract results describing geometric structures of I for which the first two steps of the above scheme can be successfully carried out is well known in the literature and the first and most popular one certainly concerns the mountain-pass geometry, introduced by Ambrosetti and Rabinowitz in their renowned paper [1]. Then, under the assumption that all the Palais-Smale sequences of I admit a strongly convergent subsequence (*Palais-Smale condition*), the minimax level c is a critical value for I (assuming I of class C^1). Unfortunately, the Palais-Smale condition fails in many concrete situations and one has to carry out the final step of the method directly, by showing that a particular Palais-Smale sequence $\{w_n\}$ can be found, which is relatively compact. A first and key point in this direction is commonly to establish the boundedness of $\{w_n\}$, which, if X is reflexive, immediately yields that (up to a subsequence) it weakly converges to some $w \in X$. Then w is a critical point if one succeeds in bringing the amount of compactness needed to conclude strong convergence, or even just in showing that $I'(\cdot)\varphi$ is weakly continuous for every fixed $\varphi \in X$ (to be precise, some more work is often required, since one usually needs $w \neq 0$).

The problem of obtaining bounded Palais-Smale sequences has been faced by several authors in different specific contexts (see, for instance, [13], [16], [19], [23], [22] and the references contained in [9]) and it has been systematically approached by Jeanjean in [9] (see also [10]-[12]), where a general method deriving from Struwe’s *monotonicity trick* [17] is presented. More precisely, the author formulates an abstract result establishing very general conditions in order that, given a family of functionals depending on a real parameter and exhibiting a “uniform” mountain-pass geometry, almost every functional of the family has a bounded Palais-Smale sequence at its mountain-pass level. Then, roughly speaking, the method consists in exploiting such a result (together with some additional compactness properties) in order to obtain a special sequence of “almost critical” points for a given functional, consisting of exact critical points of nearby functionals and thus possessing extra properties which can help in proving its boundedness. Other results in a similar abstract spirit can be found in [2], [14], [15], [18], [21], [24].

Here we present some abstract results which show that, for different geometric structures of the functional $I : X \rightarrow \mathbb{R}$ (and without any compactness assumption), the second step of the above scheme can be performed in such a way that a bounded Palais-Smale sequence is directly yielded, provided that I exhibits a suitable behaviour with respect to some sequence of continuous mappings $\psi_n : X \rightarrow X$.

Precisely, we consider a Banach space $(X, \|\cdot\|)$, a functional $I \in C^1(X, \mathbb{R})$ of the form

$$I(u) = A(u) - B(u) \quad \text{for all } u \in X \quad (1.1)$$

and we assume that:

(A) A is nonnegative and such that

$$\forall l_2 > l_1 > 0, \quad \exists \delta_{l_1, l_2} > 0, \quad \forall u \in X, \quad \text{dist}(u, A^{l_1}) \leq \delta_{l_1, l_2} \Rightarrow u \in A^{l_2}$$

(where $A^l := \{u \in X : A(u) \leq l\}$ and $\text{dist}(u, A^{l_1}) := \inf_{h \in A^{l_1}} \|u - h\|$);

(Ψ) there exists a sequence of mappings $\{\psi_n\} \subset C(X, X)$ such that $\forall n$ there exist $\alpha_n > \beta_n > 0$ satisfying

(Ψ_1) $A(u) \geq \alpha_n A(\psi_n(u))$ and $B(u) \leq \beta_n B(\psi_n(u))$ for all $u \in X$

(Ψ_2) $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 1$ and $\liminf_{n \rightarrow \infty} \frac{|1 - \beta_n|}{\alpha_n - \beta_n} < \infty$.

Observe that (A) is quite a mild assumption in practice, since in many applications one has $A(u) = (\text{const.}) \|u\|^p$, $p > 0$, so that for every $u, v \in X$ it holds

$$A(u) \leq (\text{const.}) (\|v\| + \|u - v\|)^p \rightarrow A(v) \quad \text{as } \|u - v\| \rightarrow 0.$$

Notice that in this case the existence of a bounded Palais-Smale sequence is obviously yielded by the existence of a Palais-Smale sequence on which A is bounded, but such an implication holds also true if A or $|B|$ is coercive with respect to the norm of X . Furthermore, we point out that, even when neither A nor $|B|$ is coercive with respect to the norm, the existence of a Palais-Smale sequence on which A is bounded can be a relevant information in order to get a Palais-Smale sequence which is bounded in X (see [4]).

Under the above assumptions (A) and (Ψ), we will prove that the presence of some geometric structure of minimax type is essentially sufficient in order that I exhibits, at the minimax level c , a Palais-Smale sequence on which A is bounded, i.e., a sequence $\{w_n\} \subset X$ satisfying

$$I(w_n) \rightarrow c, \quad I'(w_n) \rightarrow 0 \text{ in } X', \quad \sup_n A(w_n) < \infty. \quad (1.2)$$

This is for instance the case when I has a mountain-pass geometry, as the following theorem says.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space and $I \in C^1(X, \mathbb{R})$ a functional of the form (1.1) satisfying (A) and (Ψ). If there exist $r > 0$ and $\bar{u} \in X$ with $\|\bar{u}\| > r$ such that*

$$\inf_{\|u\|=r} I(u) > I(0) \geq I(\bar{u}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\psi_n(0)\| = \lim_{n \rightarrow \infty} \|\psi_n(\bar{u}) - \bar{u}\| = 0 \quad (1.3)$$

then there exists a sequence $\{w_n\} \subset X$ satisfying (1.2), where

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u), \quad \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = \bar{u}\}. \quad (1.4)$$

Theorem 1.1 has been already announced and used in [5] and it will be obtained in Section 2 as a consequence of a more general minimax principle (Theorem 2.1), which also allows to deduce similar results concerning geometries of saddle-point or linking type (Theorems 2.3 and 2.4). The arguments leading to such a principle derive from the ones of [3], do not employ the monotonicity trick and have been previously used in [4] and [6] within specific contexts. An estimate of the bound of $A(w_n)$ can also be obtained (cf. Remark 2.2).

This work is a preliminary version of a forthcoming paper [7], where applications to nonlinear equations without Ambrosetti-Rabinowitz type assumptions will also be given.

2 Abstract results

In this section we give our abstract results concerning the existence of Palais-Smale sequences satisfying (1.2) in the functional framework described in the Introduction. Accordingly, throughout the section we assume that $(X, \|\cdot\|)$ is a Banach space and $I \in C^1(X, \mathbb{R})$ is a functional of the form (1.1) satisfying (A) and (Ψ).

Our main result is the following.

Theorem 2.1. *Let (M, d) be a metric space and $M_0 \subset M$ a compact subspace such that there exist $\varepsilon_0 > 0$ and $\sigma \in C(M, M)$ satisfying*

$$i) \quad \sigma \text{ is uniformly continuous on } M_0^{\varepsilon_0} := \{p \in M : d(p, M_0) \leq \varepsilon_0\}$$

$$ii) \quad \sigma(M_0^{\varepsilon_0}) \subseteq M_0$$

$$iii) \quad \sigma|_{M_0} = \text{id}.$$

Let $\Gamma_0 \subset C(M_0, X)$ be such that $\bigcup_{\gamma_0 \in \Gamma_0} \gamma_0(M_0)$ is compact and define $\Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}$. Assume that

$$\sup_{\gamma_0 \in \Gamma_0} \max_{u_0 \in \gamma_0(M_0)} I(u) < c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(M)} I(u) < \infty \quad (2.1)$$

and

$$\lim_{n \rightarrow \infty} \sup_{\gamma_0 \in \Gamma_0} \max_{u \in \gamma_0(M_0)} \|\psi_n(u) - u\| = 0. \quad (2.2)$$

Then there exists a sequence $\{w_n\} \subset X$ satisfying (1.2).

Remark 2.2. According to the proof of Theorem 2.1, what we obtain exactly is that $\forall a > b := |c| \liminf_{n \rightarrow \infty} |1 - \beta_n| / (\alpha_n - \beta_n)$ the functional I has a Palais-Smale sequence $\{w_n\}$ at level c such that $A(w_n) \leq a$ for all n . Observe that it is thus likely to occur that $A(w_n) \rightarrow 0$ if $b = 0$.

Since quite long and technical, we displace the proof of Theorem 2.1 in the next section and first derive some consequences of it, beginning with the proof of Theorem 1.1.

Proof of Theorem 1.1. Set $M := [0, 1] \subset \mathbb{R}$, $M_0 := \{0, 1\}$, $\Gamma_0 := \{\gamma_0\}$ where $\gamma_0 \in C(M_0, X)$ is defined by $\gamma_0(0) := 0$ and $\gamma_0(1) := \bar{u}$, and

$$\sigma(p) := \begin{cases} 0 & \text{if } p \in [0, \varepsilon_0] \\ \frac{p - \varepsilon_0}{1 - 2\varepsilon_0} & \text{if } p \in [\varepsilon_0, 1 - \varepsilon_0] \\ 1 & \text{if } p \in [1 - \varepsilon_0, 1] \end{cases}$$

for any fixed $\varepsilon_0 \in (0, 1/2)$. So Γ and c of Theorem 2.1 coincide with the ones of (1.4) and (2.1)-(2.2) are fulfilled by (1.3). The conclusion thus follows by applying Theorem 2.1. \square

Theorem 2.3. Assume $X = Y \oplus Z$ with $\dim Y < \infty$ and assume that there exists $\rho > 0$ such that

$$\inf_{u \in Z} I(u) > \max_{u \in M_0} I(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} \max_{u \in M_0} \|\psi_n(u) - u\| = 0 \quad (2.3)$$

where $M_0 := \{u \in Y : \|u\| = \rho\}$. Then there exists a sequence $\{w_n\} \subset X$ satisfying (1.2), where

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma(M)} I(u) \\ \Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} = \text{id}\}, \quad M := \{u \in Y : \|u\| \leq \rho\}. \quad (2.4)$$

Proof. Let M and M_0 be as in Theorem 2.3 and set $\Gamma_0 := \{\gamma_0\}$ where γ_0 is the identity map of M_0 , so that Γ of Theorem 2.1 reduces to the one of (2.4). Let $\varepsilon_0 \in (0, \rho)$ and for every $u \in M$ define

$$\sigma(u) := \begin{cases} \frac{\rho}{\rho - \varepsilon_0} u & \text{if } \|u\| \leq \rho - \varepsilon_0 \\ \rho \frac{u}{\|u\|} & \text{if } \|u\| \geq \rho - \varepsilon_0. \end{cases}$$

Since standard nonretractability arguments show that $\gamma(M) \cap Z \neq \emptyset$ for every $\gamma \in \Gamma$, and thus $c \geq \inf_{u \in Z} I(u)$, assumption (2.3) imply (2.1)-(2.2) and the conclusion then follows by applying Theorem 2.1. \square

Denote $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_+^0 := [0, +\infty)$.

Theorem 2.4. Assume $X = Y \oplus Z$ with $\dim Y < \infty$ and assume that there exist $\rho > r > 0$ and $z \in Z$ with $\|z\| = r$ such that

$$\inf_{u \in Z, \|u\|=r} I(u) > \max_{u \in M_0} I(u) \quad \text{and} \quad \lim_{n \rightarrow \infty} \max_{u \in M_0} \|\psi_n(u) - u\| = 0$$

where $M_0 := \{u \in Y : \|u\| \leq \rho\} \cup \{u \in Y + \mathbb{R}_+ z : \|u\| = \rho\}$. Then there exists a sequence $\{w_n\} \subset X$ satisfying (1.2), where

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma(M)} I(u)$$

$$\Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} = \text{id}\}, \quad M := \{u \in Y + \mathbb{R}_+ z : \|u\| \leq \rho\}.$$

Proof. One proceeds as for Theorem 2.3 and the conclusion ensues from applying Theorem 2.1, provided that there exist $\varepsilon_0 > 0$ and $\sigma \in C(M, M)$ satisfying *i*), *ii*), *iii*). In order to check this property, observe that $M = \{u \in Y + \mathbb{R}_+ z : \|u\| \leq \rho\}$ is homeomorphic to a finite dimensional compact ball and, for any fixed $\bar{\varepsilon} \in (0, 1)$, define

$$\bar{\sigma}(p) := \begin{cases} \frac{1}{1-\bar{\varepsilon}}p & \text{if } |p| \leq 1-\bar{\varepsilon} \\ \frac{p}{|p|} & \text{if } |p| \geq 1-\bar{\varepsilon} \end{cases} \quad \forall p \in \bar{D} := \{p \in \mathbb{R}^{m+1} : |p| \leq 1\}$$

where $m := \dim Y + 1$. Then it is easy to see that one can take $\sigma := \phi^{-1} \circ \bar{\sigma} \circ \phi$ where $\phi : Y \oplus \mathbb{R}z \rightarrow \mathbb{R}^{m+1}$ is any homeomorphism such that $\phi(M) = \bar{D}$ and ε_0 is any radius such that $\phi(M_0^{\varepsilon_0}) \subseteq \{p \in \bar{D} : \min_{q \in \partial \bar{D}} |q - p| < \bar{\varepsilon}\}$. \square

3 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1, which will be achieved through several lemmas. Accordingly we assume all the hypotheses of the theorem and, by (2.1), fix any $\delta_* > 0$ such that

$$c_0 := \sup_{\gamma_0 \in \Gamma_0} \max_{u \in \gamma_0(M_0)} I(u) < c - \delta_*. \quad (3.1)$$

Lemma 3.1. *There exists $\varepsilon_* > 0$ such that for all $\gamma \in \Gamma$ and $u \in X$ one has*

$$\text{dist}(u, \gamma(M_0)) < \varepsilon_* \Rightarrow I(u) < c_0 + \delta_*.$$

Proof. Let $C := \{u \in X : I(u) \geq c_0 + \delta_*\}$, $K_0 := \bigcup_{\gamma_0 \in \Gamma_0} \gamma_0(M_0)$ and $\varepsilon_* := \text{dist}(C, K_0)$ ($= \inf_{u \in C, w \in K_0} \|u - w\|$). Recall that K_0 is compact by assumption and assume by contradiction $\varepsilon_* = 0$. Then there exist $\{u_n\} \subset C$ and $\{w_n\} \subset K_0$ such that $\|u_n - w_n\| \rightarrow 0$ and (up to a subsequence) $w_n \rightarrow w_0 \in K_0$, whence $\|u_n - w_0\| \leq \|u_n - w_n\| + \|w_n - w_0\| \rightarrow 0$ and thus $u_n \rightarrow w_0$. Since C is closed, this gives $w_0 \in C$, which

is a contradiction because (3.1) implies $C \cap K_0 = \emptyset$. Then the claim holds because $\text{dist}(u, \gamma(M_0)) < \varepsilon_*$ and $I(u) \geq c_0 + \delta_*$ would yield the existence of $w \in \gamma(M_0) \subseteq K_0$ (recall $\gamma|_{M_0} \in \Gamma_0$) such that $\|u - w\| < \varepsilon_*$ with $u \in C$. \square

Henceforth, we fix any $A_* > 0$ such that

$$|c| \liminf_{n \rightarrow \infty} \frac{|1 - \beta_n|}{\alpha_n - \beta_n} < A_* < \infty.$$

Moreover, passing in case to subsequences of $\{\psi_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$, by (Ψ_2) and (2.2) we assume that for all n there holds

$$\alpha_n \left(\frac{|1 - \beta_n|}{\alpha_n - \beta_n} |c| + (1 + \beta_n)(\alpha_n - \beta_n) \right) < A_* \quad (3.2)$$

and

$$\sup_{\gamma_0 \in \Gamma_0} \max_{u \in \gamma_0(M_0)} \|\psi_n(u) - u\| < \frac{\varepsilon_*}{2} \quad (3.3)$$

where $\varepsilon_* > 0$ is given by Lemma 3.1.

Lemma 3.2. *For every n and $\gamma \in \Gamma$ there exist $\varepsilon_\gamma > 0$ and $\gamma^n \in \Gamma$ such that for all $p \in M$ one has*

$$d(p, M_0) \geq \varepsilon_\gamma \Rightarrow \gamma^n(p) = \psi_n(\gamma(\sigma(p))) \quad (3.4)$$

$$d(p, M_0) < \varepsilon_\gamma \Rightarrow I(\gamma^n(p)) < c_0 + \delta_*. \quad (3.5)$$

Proof. Let $\gamma \in \Gamma$. Since γ is uniformly continuous on M_0 and σ is uniformly continuous on $M_0^{\varepsilon_0}$ with $\sigma(M_0^{\varepsilon_0}) \subseteq M_0$, $\gamma \circ \sigma$ turns out to be uniformly continuous on $M_0^{\varepsilon_0}$ and thus there exists $\varepsilon_\gamma \in (0, \varepsilon_0)$ such that $p_1, p_2 \in M_0^{\varepsilon_0}$ and $d(p_1, p_2) < \varepsilon_\gamma$ imply $\|\gamma(\sigma(p_1)) - \gamma(\sigma(p_2))\| < \varepsilon_*/2$. For every n and $p \in M$ define

$$\gamma^n(p) := \begin{cases} \psi_n(\gamma(\sigma(p))) & \text{if } d(p, M_0) \geq \varepsilon_\gamma \\ \mu(p) \psi_n(\gamma(\sigma(p))) + (1 - \mu(p)) \gamma(\sigma(p)) & \text{if } d(p, M_0) \leq \varepsilon_\gamma \end{cases}$$

where $\mu : M \rightarrow [0, 1]$ is any Uryson function such that $\mu(p) = 1$ if $d(p, M_0) = \varepsilon_\gamma$ and $\mu(p) = 0$ if $p \in M_0$. So $\gamma^n : M \rightarrow X$ is continuous and satisfies $\gamma^n|_{M_0} = \gamma \circ \sigma|_{M_0} = \gamma|_{M_0} \in \Gamma_0$, namely, $\gamma^n \in \Gamma$. Now let $p \in M$ satisfy $d(p, M_0) < \varepsilon_\gamma$ and let $\hat{p} \in M_0$ be such that $d(p, \hat{p}) < \varepsilon_\gamma$. Since $p, \hat{p} \in M_0^{\varepsilon_0}$, we deduce

$$\|\gamma(\sigma(p)) - \gamma(\hat{p})\| = \|\gamma(\sigma(p)) - \gamma(\sigma(\hat{p}))\| < \frac{\varepsilon_*}{2}$$

which gives in turn

$$\|\psi_n(\gamma(\sigma(p))) - \gamma(\hat{p})\| \leq \|\psi_n(\gamma(\sigma(p))) - \gamma(\sigma(p))\| + \frac{\varepsilon_*}{2} < \varepsilon_*$$

by (3.3). Therefore one infers $\|\gamma^n(p) - \gamma(\hat{p})\| < \varepsilon_*$ by convexity and then (3.5) follows from Lemma 3.1. \square

Corollary 3.3. *For all n one has $c \leq \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(M)} I(\psi_n(u))$.*

Proof. Set $c_n := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(M)} I(\psi_n(u))$ for brevity. Let $\delta > 0$, fix $\gamma \in \Gamma$ such that $\sup_{u \in \gamma(M)} I(\psi_n(u)) \leq c_n + \delta$ and consider $\gamma^n \in \Gamma$ given by Lemma 3.2. Then, by (3.5), (3.1) and (3.4), we have

$$\begin{aligned} c &\leq \sup_{u \in \gamma^n(M)} I(u) = \sup_{p \in M} I(\gamma^n(p)) = \sup_{p \in M, d(p, M_0) \geq \varepsilon_\gamma} I(\gamma^n(p)) \\ &= \sup_{p \in M, d(p, M_0) \geq \varepsilon_\gamma} I(\psi_n(\gamma(\sigma(p)))) \leq \sup_{p \in M} I(\psi_n(\gamma(\sigma(p)))) \\ &\leq \sup_{p \in M} I(\psi_n(\gamma(p))) = \sup_{u \in \gamma(M)} I(\psi_n(u)) \leq c_n + \delta \end{aligned}$$

which yields the claim as δ is arbitrary. \square

From now on, to any n we associate a map $\gamma_n \in \Gamma$ satisfying

$$\sup_{u \in \gamma_n(M)} I(u) \leq c + (\alpha_n - \beta_n)^2, \quad (3.6)$$

by which we define

$$\Lambda_n := \{u \in \gamma_n(M) : I(\psi_n(u)) \geq c - (\alpha_n - \beta_n)^2\}.$$

Note that $\Lambda_n \neq \emptyset$ by Corollary 3.3, which implies $\sup_{u \in \gamma_n(M)} I(\psi_n(u)) > c - (\alpha_n - \beta_n)^2$ (indeed, by continuity, Λ_n even contains an open subset of $\gamma_n(M)$).

Lemma 3.4. *For every n and $u \in \Lambda_n$ one has $A(u) \leq A_*$.*

Proof. Fix any $u \in \Lambda_n$. From (Ψ_1) it follows that

$$\begin{aligned} I(\psi_n(u)) &= A(\psi_n(u)) - B(\psi_n(u)) \leq \frac{1}{\alpha_n} A(u) - \frac{1}{\beta_n} B(u) \\ &= \frac{1}{\alpha_n} A(u) - \frac{1}{\beta_n} (A(u) - I(u)) = -\frac{\alpha_n - \beta_n}{\alpha_n \beta_n} A(u) + \frac{1}{\beta_n} I(u) \end{aligned}$$

which yields, by (3.6) and the definition of Λ_n ,

$$\frac{\alpha_n - \beta_n}{\alpha_n \beta_n} A(u) \leq \frac{1}{\beta_n} I(u) - I(\psi_n(u)) \leq \frac{1}{\beta_n} (c + (\alpha_n - \beta_n)^2) - (c - (\alpha_n - \beta_n)^2)$$

and thus

$$\begin{aligned} A(u) &\leq \alpha_n \left(\frac{1 - \beta_n}{\alpha_n - \beta_n} c + (1 + \beta_n)(\alpha_n - \beta_n) \right) \\ &\leq \alpha_n \left(\frac{|1 - \beta_n|}{\alpha_n - \beta_n} |c| + (1 + \beta_n)(\alpha_n - \beta_n) \right) < A_* \end{aligned}$$

by (3.2). \square

Remark 3.5. By suitably changing assumption (3.2), the above proof of Lemma 3.4 shows that neither that case $c > 0$ and $\liminf_{n \rightarrow \infty} (1 - \beta_n) / (\alpha_n - \beta_n) < 0$, nor the case $c < 0$ and $\limsup_{n \rightarrow \infty} (1 - \beta_n) / (\alpha_n - \beta_n) > 0$ can occur under the assumed hypotheses, since this would yield the contradiction $A(u) < 0$ for all $u \in \Lambda_n$ ($\neq \emptyset$) with n large enough.

Lemma 3.6. *One has $\lim_{n \rightarrow \infty} \sup_{u \in \Lambda_n} |I(u) - c| = \lim_{n \rightarrow \infty} \sup_{u \in \Lambda_n} |I(\psi_n(u)) - I(u)| = 0$.*

Proof. Since $A \geq 0$, for every $u \in \Lambda_n$ from (Ψ_1) one deduces

$$I(\psi_n(u)) \leq \frac{1}{\alpha_n} A(u) - \frac{1}{\beta_n} B(u) = -\frac{\alpha_n - \beta_n}{\alpha_n \beta_n} A(u) + \frac{1}{\beta_n} I(u) \leq \frac{1}{\beta_n} I(u)$$

so that, by the definition of Λ_n , one has

$$I(u) - c \geq \beta_n I(\psi_n(u)) - c \geq \beta_n (c - (\alpha_n - \beta_n)^2) - c.$$

By (3.6) this yields $\sup_{u \in \Lambda_n} |I(u) - c| \rightarrow 0$ as $n \rightarrow \infty$. By (3.6) and the definition of Λ_n one also has $I(u) - I(\psi_n(u)) \leq 2(\alpha_n - \beta_n)^2$ for all $u \in \Lambda_n$, whereas (Ψ_1) and Lemma 3.4 yield

$$\begin{aligned} I(\psi_n(u)) - I(u) &= A(\psi_n(u)) - A(u) + B(\psi_n(u)) + B(u) \\ &\leq \frac{|1 - \alpha_n|}{\alpha_n} A(u) + \frac{|1 - \beta_n|}{\beta_n} |B(u)| \\ &\leq \frac{|1 - \alpha_n|}{\alpha_n} A_* + \frac{|1 - \beta_n|}{\beta_n} (A(u) + |I(u)|) \\ &\leq \left(\frac{|1 - \alpha_n|}{\alpha_n} + \frac{|1 - \beta_n|}{\beta_n} \right) A_* + \frac{|1 - \beta_n|}{\beta_n} \left(c + \sup_{u \in \Lambda_n} |I(u) - c| \right). \end{aligned}$$

Hence $\sup_{u \in \Lambda_n} |I(\psi_n(u)) - I(u)| \rightarrow 0$ as $n \rightarrow \infty$. \square

Now fix $l_1, l_2 > 0$ such that $A_* < l_1 < l_2$. For every $k \geq 1$, define

$$U_k := \left\{ u \in X : A(u) \leq l_2 + \frac{1}{k}, \quad |I(u) - c| \leq \frac{1}{k} \right\}$$

and, by (Ψ_2) and Lemma 3.6, take $n_k \in \mathbb{N}$ such that

$$\alpha_{n_k} > \frac{A_*}{l_1}, \quad \sup_{u \in \Lambda_{n_k}} I(\psi_{n_k}(u)) \leq c + \frac{1}{16k}, \quad \sup_{u \in \Lambda_{n_k}} |I(u) - c| \leq \frac{1}{k}. \quad (3.7)$$

Hence $\Lambda_{n_k} \subseteq U_k$ for all k and U_k is not empty.

In order to conclude the proof of Theorem 2.1 we need to apply a well known deformation lemma (see [20, Lemma 2.3]), which, for completeness, we recall here for the space X , our functional I and its minimax level c .

Lemma 3.7. *Let $\mathcal{S} \subset X$ and $\varepsilon, \delta > 0$ be such that $\|I'(u)\|_{X'} \geq 8\varepsilon/\delta$ for all $u \in \mathcal{S}_{2\delta}$ satisfying $|I(u) - c| \leq 2\varepsilon$, where $\mathcal{S}_{2\delta} := \{v \in X : \text{dist}(v, \mathcal{S}) \leq 2\delta\}$. Then there exists $\eta \in C([0, 1] \times X, X)$ such that*

- $\eta(\tau, \cdot)$ is a homeomorphism of X for every $\tau \in [0, 1]$
- $\eta(\tau, u) = u$ provided that $\tau = 0$ or $|I(u) - c| > 2\varepsilon$ or $u \notin \mathcal{S}_{2\delta}$
- $I(\eta(1, u)) \leq c - \varepsilon$ provided that $I(u) \leq c + \varepsilon$ and $u \in \mathcal{S}$
- $I(\eta(\cdot, u))$ is nonincreasing for every $u \in X$.

Proof of Theorem 2.1. Consider the sublevels A^{l_1}, A^{l_2} of A and, by hypothesis (A), fix $\delta_0 > 0$ such that

$$\forall u \in X \quad \text{dist}(u, A^{l_1}) \leq \delta_0 \Rightarrow u \in A^{l_2}. \quad (3.8)$$

Then assume by contradiction that

$$\exists \bar{k} > \max \left\{ \frac{1}{\delta_0^2}, \frac{1}{8(c - c_0 - \delta_*)} \right\} \quad \forall u \in U_{\bar{k}} \quad \|I'(u)\|_{X'} \geq \frac{1}{\sqrt{\bar{k}}}. \quad (3.9)$$

Recall that $c - c_0 - \delta_* > 0$ by (3.1). We apply Lemma 3.7 with $\mathcal{S} = A^{l_1}$, $\varepsilon = 1/16\bar{k}$ and $\delta = 1/2\sqrt{\bar{k}}$ (so that $8\varepsilon/\delta = 1/\sqrt{\bar{k}}$). Observe that if $u \in \mathcal{S}_{2\delta}$ satisfies $|I(u) - c| \leq 1/8\bar{k}$ then $u \in U_{\bar{k}}$ (and thus the last inequality of (3.9) holds), since $2\delta = 1/\sqrt{\bar{k}} < \delta_0$ and (3.8) applies. So there exists a homeomorphism $\Phi : X \rightarrow X$ (namely $\Phi := \eta(1, \cdot)$ of Lemma 3.7) such that

- (i) $\Phi(u) = u$ if $|I(u) - c| \geq c - c_0 - \delta_*$ (recall $c - c_0 - \delta_* > \frac{1}{8\bar{k}} = 2\varepsilon$)
- (ii) $I(\Phi(u)) \leq c - \frac{1}{16\bar{k}}$ if $A(u) \leq l_1$ and $I(u) \leq c + \frac{1}{16\bar{k}}$
- (iii) $I(\Phi(u)) \leq I(u)$ for every $u \in X$,

by which we define the mapping $\gamma := \Phi \circ \gamma_{n_{\bar{k}}}^{n_{\bar{k}}} \in C(M, X)$, where $\gamma_{n_{\bar{k}}}^{n_{\bar{k}}} = (\gamma_{n_{\bar{k}}})^{n_{\bar{k}}} \in \Gamma$ is the mapping associated to $\gamma_{n_{\bar{k}}}$ by Lemma 3.2. Then, by (3.5) of Lemma 3.2, $p \in M$ and $d(p, M_0) < \varepsilon_{\gamma_{n_{\bar{k}}}}$ imply $I(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)) < c_0 + \delta_*$, which yields

$$\left| I(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)) - c \right| = c - I(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)) > c - c_0 - \delta_*$$

and so, by (i), $\gamma(p) = \Phi(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)) = \gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)$ and $I(\Phi(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p))) = I(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)) < c_0 + \delta_* < c$. Hence, in particular, $\gamma|_{M_0} = \gamma_{n_{\bar{k}}}^{n_{\bar{k}}}|_{M_0} \in \Gamma_0$ and thus $\gamma \in \Gamma$. Moreover, by

(3.4), one has

$$\begin{aligned}
\sup_{u \in \gamma(M)} I(u) &= \sup_{p \in M} I\left(\Phi\left(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)\right)\right) = \sup_{p \in M, d(p, M_0) \geq \varepsilon \gamma_{n_{\bar{k}}}} I\left(\Phi\left(\gamma_{n_{\bar{k}}}^{n_{\bar{k}}}(p)\right)\right) \\
&= \sup_{p \in M, d(p, M_0) \geq \varepsilon \gamma_{n_{\bar{k}}}} I\left(\Phi\left(\psi_{n_{\bar{k}}}(\gamma_{n_{\bar{k}}}(\sigma(p)))\right)\right) \\
&\leq \sup_{p \in M} I\left(\Phi\left(\psi_{n_{\bar{k}}}(\gamma_{n_{\bar{k}}}(\sigma(p)))\right)\right) \\
&\leq \sup_{u \in \gamma_{n_{\bar{k}}}(M)} I\left(\Phi\left(\psi_{n_{\bar{k}}}(u)\right)\right)
\end{aligned} \tag{3.10}$$

and this yields a contradiction. Indeed, on one hand, if $u \in \gamma_{n_{\bar{k}}}(M) \setminus \Lambda_{n_{\bar{k}}}$, by (iii) and the definition of $\Lambda_{t_{\bar{m}}}$ we get

$$I\left(\Phi\left(\psi_{n_{\bar{k}}}(u)\right)\right) \leq I\left(\psi_{n_{\bar{k}}}(u)\right) < c - (\alpha_{n_{\bar{k}}} - \beta_{n_{\bar{k}}}).$$

On the other hand, if $u \in \Lambda_{n_{\bar{k}}}$, we have $I(\psi_{n_{\bar{k}}}(u)) \leq c + 1/16\bar{k}$ by (3.7) and $A(\psi_{n_{\bar{k}}}(u)) \leq A(u)/\alpha_{n_{\bar{k}}} < l_1$ by (Ψ_1) , (3.7) and Lemma 3.4, so that (ii) gives

$$I\left(\Phi\left(\psi_{n_{\bar{k}}}(u)\right)\right) \leq c - \frac{1}{16\bar{k}}.$$

Therefore, by (3.10), one obtains $\sup_{u \in \gamma(M)} I(u) < c$ and the definition (2.1) of c is contradicted. \square

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